Comparative Analysis of Nuclear Magnetic Resonance and Whole Angle Coriolis Vibratory Gyroscopes

Igor P. Prikhodko, Alexander A. Trusov, Andrei M. Shkel
MicroSystems Laboratory, University of California, Irvine, CA, USA
e-mails: {iprikhod, atrusov, ashkel}@uci.edu

***Abstract*—This paper's focus is a theoretical comparison of Nuclear Magnetic Resonance Gyroscope (NMRG) and Whole-Angle (WA) Coriolis Vibratory Gyroscopes (CVG) dynamics, showing the parallels in structure of basic equations and deriving performance scaling laws. The comparison relies on the method of averaging to derive the linearized equations of motion and analytical solutions for both types of sensors. While equations for WA CVG in terms of slow-varying components have been extensively published in literature, the contribution of this work is derivation of the linearized equations of motion for NMRG using the method of averaging, as well as its comparison to the WA CVG dynamics. Finally, the paper identifies "optimization knobs" for the two types of sensors (e.g., ring-down time for CVG and relaxation time for NMRG) and discusses similarities and dissimilarities in their dynamics.**

***Keywords*—nuclear magnetic resonance gyroscope, Coriolis vibratory gyroscope, method of averaging**

I. Introduction

To address the challenges associated with the long-term drift of chip-scale inertial sensors, fusion of sensors with dissimilar, but complementary physics was recently proposed [1]. In this context, Nuclear Magnetic Resonance Gyroscopes (NMRGs) [2] and Whole-Angle (WA) Coriolis Vibratory Gyroscopes (CVGs) [3,4] exhibit complementary dynamics, making these sensors attractive for comparative analysis. Toward this goal, an in-depth analysis of their dynamics and performance scaling laws are presented in this paper.

The comparison relies on the method averaging to derive the linearized equations of motion and analytical solutions for both types of sensors. The WA mode of CVG operation was chosen for the comparison with NRMG since in this case the output of both sensors is the angle. While equations for WA CVG in terms of slow-varying components have been published in literature [4], the contribution of this work is derivation of the linearized equations of motion for NMRG using method of averaging, followed by comparative analysis.

II. Theoretical Model for NMRG

NMRG measures angle or angular rate by observing the response of the hyperpolarized noble gas to the rotation. Here we focus on the NMRG operated in an angle-output mode, – a realization implemented in [2] (rate output is possible when NMRG is operated in force-to-rebalance mode). The dynamics of the noble gas nuclei spin is described by Bloch equations in terms of magnetization components M_x, M_y, and M_z, Table I.

This work was supported in partially by the DARPA/AMRDEC under Contract W31P4Q-13-1-0008 and NSF grant CMMI-0928999.

TABLE I. NMRG Dynamics for Angle Output Mode

Equations of motion:

$$\ddot{M}_y + \left(\frac{2}{T_2} - k_P\gamma M_z\right)\dot{M}_y + \left((\omega_0-\Omega)^2 + \frac{1}{T_2^2} - k_P\gamma\left[\dot{M}_z + \frac{M_z}{T_2}\right]\right)M_y = 0,$$

$$\dot{M}_z = -k_P\gamma M_y^2 + \frac{1}{T_1}(M_0 - M_z).$$

Steady-state solution (for $\omega_0^2 >> 1/T_2^2$):

$$M_y = \frac{1}{k_P\gamma}\sqrt{\frac{2(k_P\gamma M_0 T_2 - 2)}{T_1 T_2}}\cos\left[\omega_0 t - \int\Omega_z dt + \theta_0\right]$$

M_y	y component of magnetization	M_z	z component of magnetization
T_1	longitudinal relaxation time	T_2	transverse relaxation time
ω_0	Larmor frequency	M_0	M_z equilibrium value
B_0	z-axis constant magnetic field	B_1	x-axis magnetic field, $B_1=k_P\cdot M_y$
ω	observed frequency: $\omega=\gamma B_0+\Omega_z=\omega_0+\Omega_z$	k_P	feedback gain
		θ_0	initial phase
γ	gyromagnetic ratio	Ω_z	angular rate

TABLE II. CVG Dynamics for Whole-Angle Mode

Equations of motion [4]:

$$\ddot{x} + \frac{2}{\tau_x}\dot{x} + \left(\omega_x^2 - k^2\Omega^2\right)x = f_x + \left[2k\Omega_z - \Delta(1/\tau)\sin 2\theta_\tau\right]\dot{y} + \left[\Delta\omega\sin 2\theta_\omega + k\dot{\Omega}\right]y,$$

$$\ddot{y} + \frac{2}{\tau_y}\dot{y} + \left(\omega_y^2 - k^2\Omega^2\right)y = f_y - \left[2k\Omega_z + \Delta(1/\tau)\sin 2\theta_\tau\right]\dot{x} + \left[\Delta\omega\sin 2\theta_\omega - k\dot{\Omega}\right]x.$$

Steady-state solution for an ideal case ($f_x = f_y = 0$, $\omega = \omega_x = \omega_y$, $\tau_x = \tau_y \to \infty$):

$$x = a_0\cos(-k\textstyle\int\Omega_z dt)\cos(\omega t + \varphi)$$

$$y = a_0\sin(-k\textstyle\int\Omega_z dt)\cos(\omega t + \varphi).$$

x	x-axis (drive-mode) displacement	y	y-axis (sense-mode) displacement
τ_x	x-axis ring-down time constant	τ_y	y-axis ring-down time constant
ω_x	drive-mode natural frequency	ω_y	sense-mode natural frequency
f_x	amplitude force along x-axis	f_y	amplitude force along y-axis
a_0	drive-mode amplitude	φ	initial phase
k	angular gain factor	Ω_z	angular rate
$\Delta\omega$	frequency mismatch	$\Delta(1/\tau$	damping mismatch
θ_τ	angle of principal axis of damping	θ_ω	principal axis of elasticity angle

To sustain oscillations (nuclei spin precession) in NMRG, the output signal M_y is fed back to x-axis magnetic coils with a proportional gain k_P, i.e. $B_1 = k_P\cdot M_y$ (in addition to constant z-axis field B_0), so that equations of motion can be written in the form shown in Table I. To draw a parallel with the WA CVG (Table II), the Bloch equations are represented in the form of first- and second-order differential equations (as opposed to the system of first-order equations). Despite the apparent non-linearity, the form of the equations reveals that the solution is periodic, and the method of averaging [5] can be applied to derive the analytical solution. Using the method of averaging [5] presented for CVG [4], we derive linearized equations for NRMG in a similar fashion.

Assuming the solution M_y is periodic, $M_y = A\cdot\cos(\lambda t + \theta)$ with amplitude A, phase θ, and frequency $\lambda = \sqrt{(\omega_0^2 + 1/T_2^2)}$, the equations of motion for NMRG in Table I transforms to:

$$\dot{A} = -\left(\frac{2}{T_2} - \gamma k_P M_z\right)\frac{A}{2},$$
$$\dot{M}_z = -\frac{\gamma k_P}{2}A^2 + \frac{1}{T_1}(M_0 - M_z), \quad (1)$$
$$\dot{\theta} = -\frac{\gamma k_P}{2\lambda}\left(\lambda\dot{M}_z + \frac{1}{T_2}M_z\right) - \Omega,$$

which constitute a system of three first-order equations for slow-varying variables A, θ, and M_z, describing the closed-loop NMRG dynamics. The steady-state solution ($dA/dt = = dM_z/dt = 0$) of the first two equations of system (1) is:

$$A = \frac{1}{\gamma k_P}\sqrt{\frac{2(M_0\gamma k_P T_2 - 2)}{T_1 T_2}}, \; M_z = \frac{2}{\gamma k_P T_2}. \quad (2)$$

The stability analysis shows that the solution (2) corresponds to stable harmonic oscillations (self-resonance) as long as:

$$\frac{2}{T_2} - M_0\gamma k_P < 0, \quad (3)$$

which can be satisfied by choosing a proper feedback gain k_P. Substituting solution (2) into the last equation of system (1) and assuming $\omega_0^2 >> 1/T_2^2$ (valid for typical relaxation time $T_2 = 30$ s and Larmor frequency $\omega_0 = 300$ Hz$\cdot 2\cdot\pi$), we obtain:

$$\theta = -\int \Omega dt + \theta_0, \quad (4)$$

showing that the NMRG is a rate integrating gyroscope with a scale factor of 1. As can be seen from (1), the angle drift in NMRG could be caused by the instability in M_z (e.g. caused by z-axis B_0 field fluctuation), inseparable from the true angle.

III. THEORETICAL MODEL FOR WHOLE-ANGLE CVG

Although the dynamics of CVG in terms of slow-varying components is presented in [4], we summarize the main results to perform comparative analysis of NMRG and WA CVG. The solution of equations in Table II is periodic, and it can be represented in terms of slow-varying components c_x, c_y, s_x, s_y:

$$x = c_x\cos(\omega t + \phi) + s_x\sin(\omega t + \phi),$$
$$y = c_y\cos(\omega t + \phi) + s_y\sin(\omega t + \phi). \quad (5)$$

By substituting (6) into the equations of motion in Table II and using method of averaging, the steady-state solution is [4]:

$$c_x = a\cos\theta, \qquad c_y = a\sin\theta, \quad (6)$$
$$s_x = a\cos\theta\delta\varphi - q\sin\theta, \quad s_y = a\sin\theta\delta\varphi + q\cos\theta, \quad (7)$$

where a, q, θ, $\delta\varphi$ are amplitude, quadrature, precession angle, phase error, respectively. In ideal case $a >> q$, $\delta\varphi << 1$ we get:

$$\theta = -k\int \Omega dt + \theta_0, \quad (8)$$

showing that WA CVG is a rate integrating gyroscope with a scale factor of k. In presence of demodulation phase error $\delta\varphi$ the angle θ is derived using combination of c_x, c_y, s_x, s_y [4]:

$$\frac{1}{2}\arctan\frac{2(c_x c_y + s_x s_y)}{c_x^2 + s_x^2 - c_y^2 + s_y^2} = \frac{1}{2}\arctan\frac{\sin 2\theta}{\cos 2\theta} = \theta. \quad (9)$$

Despite the fact that WA CVG is independent of the phase error, it is still susceptible to damping/stiffness mismatch [4]:

$$\dot{\theta} = -k\Omega + \frac{1}{2}\Delta\left(\frac{1}{\tau}\right)\sin 2(\theta - \theta_\tau) + \Delta\omega\cos 2(\theta - \theta_\tau)\frac{q}{a}, \quad (10)$$

showing angle drift for CVG (notations explained in Table II).

IV. COMPARATIVE ANALYSIS: PERFOMANCE SCALING LAWS

Besides dissimilar noise mechanisms, the parallels can be drawn between NRMG and WA CVG. Both sensors rely on phase measurements for inertial rotation detection. They exhibit theoretically unlimited input rate range and wide measurement bandwidth [4,6]. Specifically, bandwidth and range are limited by the natural frequency for WA CVG [4], and by Xenon Larmor frequency for NMRG* [6]. The key parameters defining the AWN and ARW for WA CVG are Q-factor and ring-down time [4], and a transverse relaxation time constants for NMRG [6]. The mass for CVG and the number of polarized atoms for NMRG define the SNR [6]. Finally, the short-term noise limit is defined by SNR for both NMRG and WA CVG.

V. CONCLUSIONS

The analysis based on method of averaging revealed that both NMRG and WA CVGs are phase modulated gyroscopes, i.e. the phase of their output signals is a measure of the rotation angle. The error analysis of NMRG showed that the instability of a z-axis component of magnetization vector is a primary source of angle drift (for particular feedback system presented here). The error analysis of WA CVG showed that the angle readout method is robust to variations between the gyroscope phase and the phase of an external clock (PLL), but susceptible to the frequency and damping mismatches, which, in turn, contribute to the angle drift. Finally, we theoretically demonstrated that both types of sensors exhibiting different but complementary strengths and weaknesses in terms of bandwidth, range, and sensitivity, but having dissimilar angle drift mechanisms.

VI. ACKNOWLEDGMENT

Authors would like to thank Dr. Larsen from Northrop Grumman and Dr. Zotov from UCI for insightful discussions.

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* In practice, the bandwidth of NRMG is limited by the bandwidth of the magnetic field readout system, e.g. the alkali magnetometer.